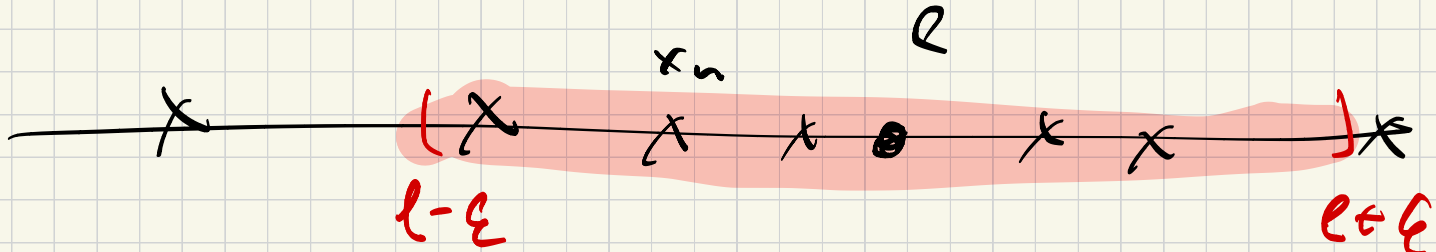
Analysis I Lecture 10

Last time:

<u>limit of the sequence:</u>

$\lim_{n\to\infty} x_n = \ell$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$

s.t. $\forall n > N$ $|x_n - \ell| < \varepsilon$



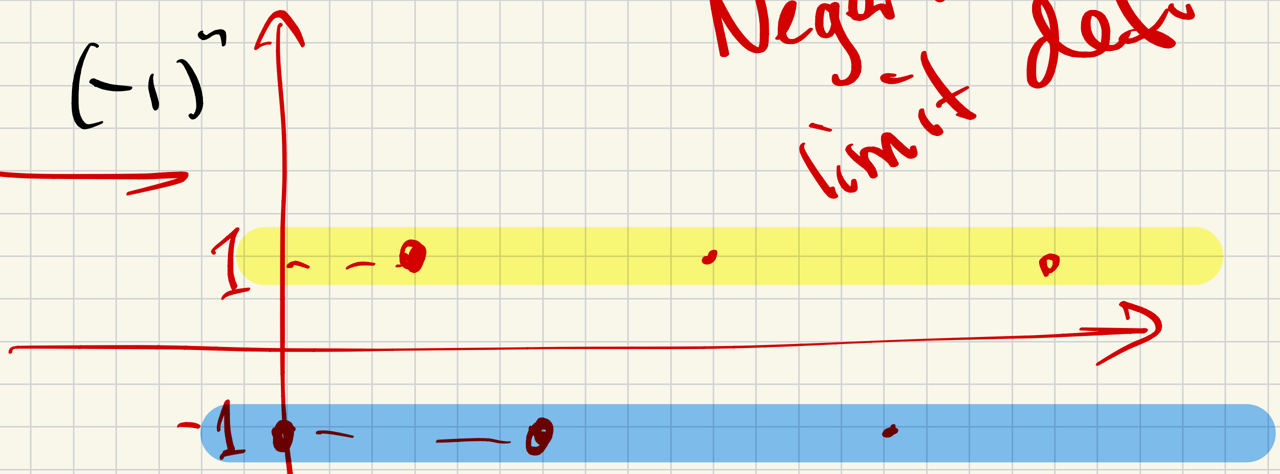
(x_n) diverges if it doesn't have a limit:

$\forall \ell \in \mathbb{R}$: $\exists \varepsilon > 0$ s.t.

$\forall N \in \mathbb{N}$ $\exists n > N$ with $|x_n - \ell| > \varepsilon$.

Negation of limit definition

e.g. $x_n = (-1)^n$



Limits algebra

Proposition Let (x_n), (y_n) be two convergent sequences with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ then

1. Sequence $(x_n + y_n)$ is convergient and $\lim_{n\to\infty}(x_n + y_n) = x + y$
2. Sequence $(x_n \cdot y_n)$ is convergient and $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$
3. If also $y \neq 0$, sequence $\left(\frac{x_n}{y_n}\right)$ is convergient and $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$
4. If $\exists N \in \mathbb{N}$ s.t. $x_n \leq y_n \ \forall n > N$ then $x \leq y$.

Example Let $\alpha, \beta \in \mathbb{R}$

and $\lim_{n\to\infty} x_n = x$; $\lim_{n\to\infty} y_n = y$ then

$$\lim_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha x + \beta y$$

if $\alpha = 1$, $\beta = -1$:

In particular, $\lim_{n\to\infty} (x_n - y_n) = x - y$

Example

It is Not true that if $(x_n + y_n)$ converges then both (x_n) and (y_n) converge.

$$\underbrace{(-1)^n}_{x_n} + \underbrace{(-1)^{n+1}}_{y_n} = (-1)^n - (-1)^n = \underbrace{0}_{x_n + y_n}$$

$\lim_{n \to \infty} x_n$ or $\lim_{n \to \infty} y_n$ does not exists

$\lim_{n \to \infty} x_n + y_n = 0$

However, if 2 out 3 sequences

(x_n), (y_n), $(x_n + y_n)$ converge

then the third one also

converges.

Example $\frac{1}{n} + \frac{1}{n^2} = x_n = \frac{n+1}{n^2}$

polynomials

We know that $\lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$

then we know $\lim_{n\to\infty} \frac{1}{n} + \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n^2} =$

$= 0.$

More generally, we can analize limits of sequences defined as fractions of two polynomials.

For example:

$$x_n = \frac{n^2 + 2n + 3}{4n^2 + 5n + 6} = \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{4 + \frac{5}{n} + \frac{6}{n^2}}$$

$$x_n = \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{4 + \frac{5}{n} + \frac{6}{n^2}}$$ if converge

then
$$\lim_{n\to\infty} x_n = \frac{\lim_{n\to\infty}\overbrace{\left(1 + \frac{2}{n} + \frac{3}{n^2}\right)}^{y_n}}{\lim_{n\to\infty}\underbrace{\left(4 + \frac{5}{n} + \frac{6}{n^2}\right)}_{z_n}}$$

So it is enough to analyze sequences (y_n) and (z_n)

Start with $y_n = 1 + \frac{2}{n} + \frac{3}{n^2}$

Since $\lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty} \frac{2}{n} = 0$

$\lim_{n\to\infty} \frac{3}{n^2} = 0$

we get $\lim_{n\to\infty} y_n = 1 + 0 + 0 = 1$

Similarly
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} 4 + \frac{5}{n} + \frac{6}{n^2}$$

$$= \underbrace{\lim_{n\to\infty} 4}_{4} + \underbrace{\lim_{n\to\infty} \frac{5}{n}}_{0} + \underbrace{\lim_{n\to\infty} \frac{6}{n^2}}_{0} = 4$$

And
$$\lim_{n\to\infty} x_n = \frac{\lim_{n\to\infty} y_n}{\lim_{n\to\infty} z_n} = \frac{1}{4}$$

In general, $x_n = \dfrac{p(n)}{q(n)}$ with

$$p(n) = a_s n^s + a_{s-1} n^{s-1} + \cdots + a_0$$

$$q(n) = b_t n^t + b_{t-1} n^{t-1} + \cdots + b_0$$

then if $s < t$, $\lim\limits_{n\to\infty} x_n = 0$

if $s = t$, $\lim\limits_{n\to\infty} x_n = \dfrac{a_s}{b_t}$

if $s > t$, then $\lim\limits_{n\to\infty} x_n = \pm\infty$

Definition We say that (x_n) approaches $+\infty$ (or $-\infty$) denoted by $\lim_{n\to\infty} x_n = +\infty$ (or $\lim_{n\to\infty} x_n = -\infty$) if

$$\forall\, C \in \mathbb{R}_{>0} \quad \exists\, N \in \mathbb{N} \quad \text{s.t.}$$

$$\forall\, n > N \qquad x_n > C \quad (\text{or } x_n < -C).$$

Back to $x_n = \dfrac{p(n)}{q(n)}$

If $\overset{s}{\overset{\shortparallel}{\deg}}(p) > \overset{t}{\overset{\shortparallel}{\deg}}(q)$ then

$\lim_{n\to\infty} x_n = \pm\infty$ a_s/b_t

depending on the sign of

the ratio of top coeff. of $p(n)$ and $q(n)$

Example

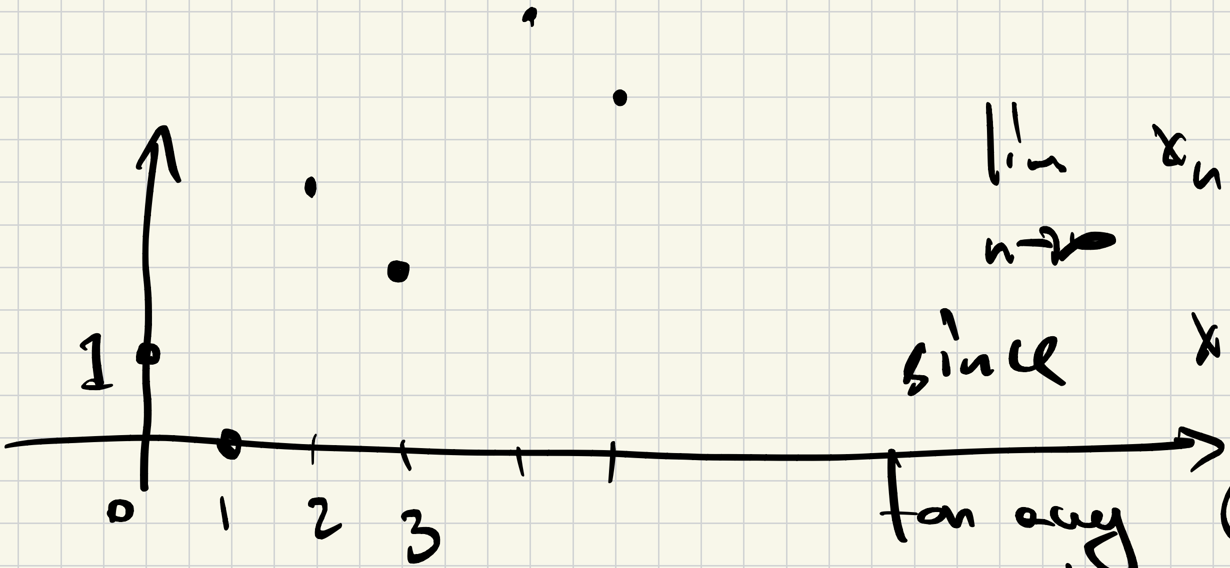
$$x_n = n = \frac{\overset{p(n)}{n}}{\underset{q(n)}{1}}$$ polynomials

$\deg(p(n)) = 1 > \deg(q(n)) = 0$

$\Rightarrow \quad \lim_{n \to \infty} x_n = +\infty$

Example $x_n = n + (-1)^n$

Sequence x_n is not monotone



$$\lim_{n\to\infty} x_n = +\infty$$

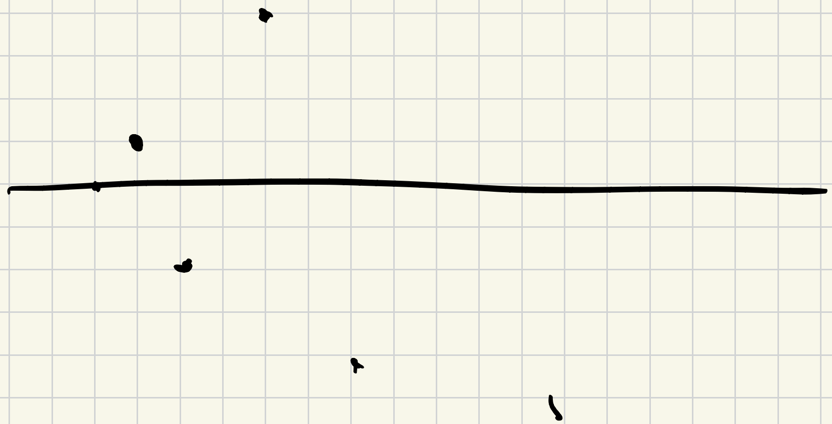
since $x_n \geq n-1$

for any C $\exists N$ s.t. $N-1>C$

and in particular $x_n \geq C$ $\forall n>N$

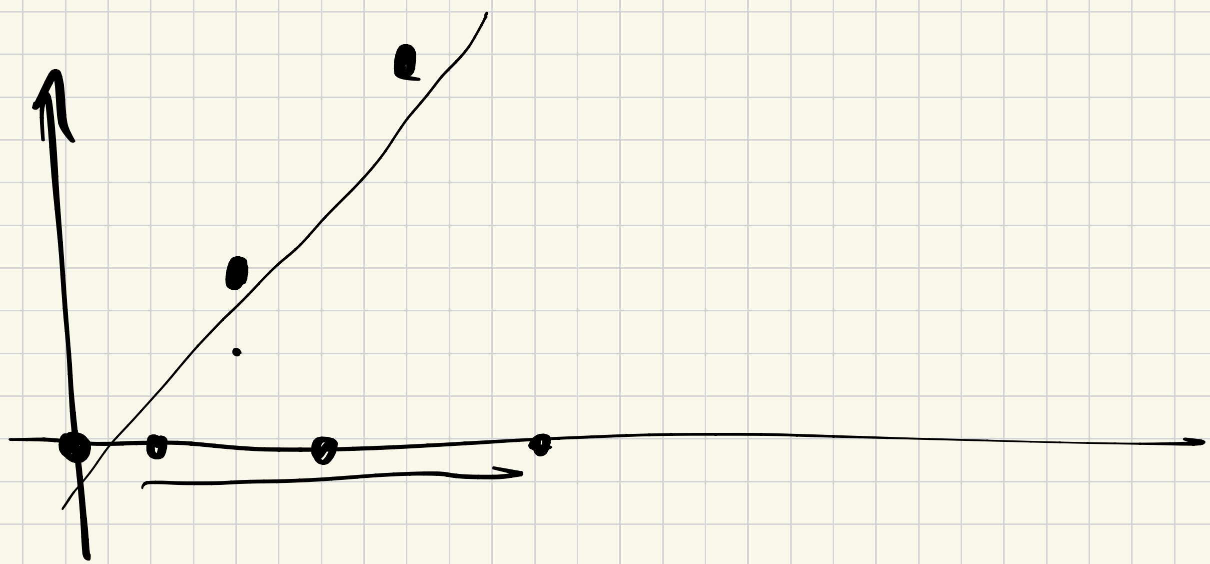
Caution! Not enough to be unbounded to approach $\pm\infty$!

Example $x_n = (-1)^n \cdot n$ it doesn't approach $\pm\infty$.



Another example:

$$x_n = \begin{cases} n & \text{for } n\text{-even} \\ 0 & \text{otherwise} \end{cases}$$



Algebra of infinite limits

Proposition Let (x_n), (y_n) be two sequences.

1) Assume $\lim\limits_{n\to\infty} x_n = +\infty$ and (y_n) bounded from below. Then,

(i) $\lim\limits_{n\to\infty} x_n + y_n = +\infty$

(ii) if $\exists\, A \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ s.t. $\forall n \geq N$ $y_n \geq A$, then $\lim\limits_{n\to\infty} y_n \cdot x_n = +\infty$

(iii) if (y_n) is bounded then $\lim\limits_{n\to\infty} \frac{y_n}{x_n} = 0$

2) Assume $\lim_{n\to\infty} x_n = -\infty$ and (y_n) bounded from above. Then,

(i) $\lim_{n\to\infty} x_n + y_n = -\infty$

(ii) if $\exists A \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ s.t. $\forall n \geq N$ $y_n \geq A$, then $\lim_{n\to\infty} y_n \cdot x_n = -\infty$

(iii) if (y_n) is bounded then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$

Examples

1) $z_n = \underbrace{(-1)^n}_{y_n} + \underbrace{n}_{x_n}$

$\lim_{n\to\infty} x_n = +\infty$

y_n is bounded below

$\Rightarrow \lim_{n\to\infty} z_n = +\infty$

Non-example

$x_n = n \qquad y_n = -n+1$

then $\lim x_n = +\infty$ but y_n is not bounded from below!

And $(x_n + y_n) = n - n + 1 = 1 \qquad \lim_{n\to\infty}(x_n + y_n) \neq +\infty$

<u>Example</u> (ii) Non-example:

$$x_n = n, \quad y_n = (-1)^n$$

Note $\lim_{n\to\infty} x_n = +\infty$ but $\nexists$ $A \in \mathbb{R}_{>0}$

s.t. $y_n > A$ for any $n > N$.

So we can't use the prop. Indeed $x_n \cdot y_n = (-1)^n n$ does not tend to $+\infty$.

Example for (iii) take $x_n = n$

$y_n = (-1)^n$

then $\lim_{n\to\infty} x_n = +\infty$ and y_n is bounded

Note that y_n doesn't converge.

So by (iii) we get,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \frac{(-1)^n}{n} = 0.$$

Careful with $\lim x_n = \lim y_n = +\infty$!

- $x_n = n \qquad y_n = n^2$ then

$$\frac{x_n}{y_n} = \frac{1}{n} \quad \text{so} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = 0$$

- $x_n = n^2 \qquad y_n = n$

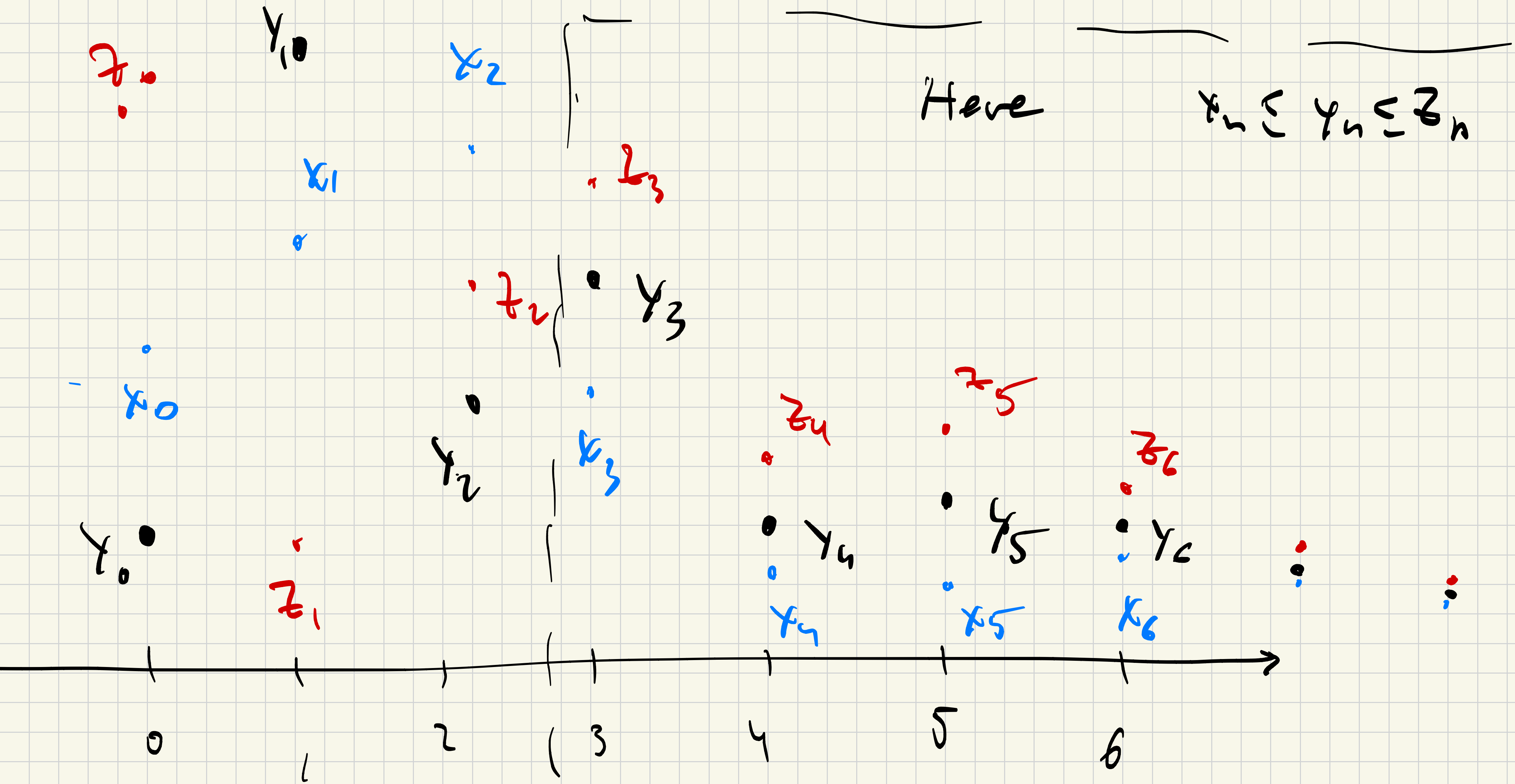
$$\lim \frac{x_n}{y_n} = \lim \frac{n^2}{n} = \lim n = +\infty$$

Squeeze theorem (two policemens theorem)

Theorem Let (x_n), (z_n) be two convergent sequences s.t. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} (z_n) = a \in \mathbb{R}$

Let (y_n) be such that $\exists N$ with $\forall n > N$ $x_n \le y_n \le z_n$ then

y_n converges and $\lim_{n\to\infty} y_n = a$.



Here $x_n \le y_n \le z_n$

Example $\lim_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{\sqrt{n}} = 1$

indeed $\underbrace{1}_{x_n} \leq 1 + \frac{1}{n} + \frac{1}{\sqrt{n}} \leq \underbrace{1 + \frac{2}{\sqrt{n}}}_{z_n}$

$\lim_{n\to\infty} x_n = 1 = 1 + 0 = \lim_{n\to\infty} z_n$

$\Rightarrow \lim_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{n^2} = 1$

we interested in

Corollary Let (y_n) be a sequence, $l \in \mathbb{R}$ and (a_n) be a sequence of positive numbers s.t.

service sequence

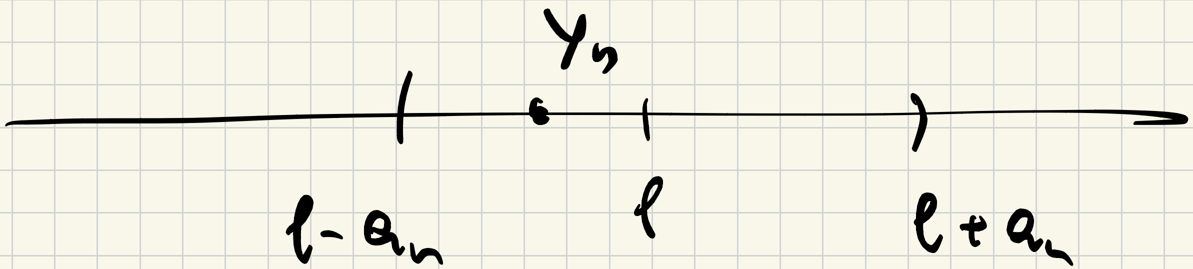
1) $\lim_{n\to\infty} a_n = 0$

2) $\exists N$ s.t. $\forall n > N$ $|y_n - l| < a_n$

then $\lim_{n\to\infty} y_n = l$

Sketch of proof:

If $|y_n - \ell| < a_n$ then $\ell - a_n < y_n < \ell + a_n$



So $x_n = \ell - a_n$ and $z_n = \ell + a_n$ then

1) $$\lim_{n\to\infty} x_n = \underbrace{\lim \ell}_{=\ell} - \underbrace{\lim a_n}_{=0} = \ell = \lim_{n\to\infty} z_n$$

2). Moreover, for any $n > N$ we have

$$x_n \le y_n < z_n$$

$\Rightarrow$

by Squeeze thm $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = \ell$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$$

$\blacksquare$

<u>Corollary</u> Let (x_n) be convergent sequence with $\lim_{n\to\infty} x_n = 0$ then if (y_n) is bounded, then $(x_n \cdot y_n)$ converges and $\lim_{n\to\infty} (x_n \cdot y_n) = 0$.

Thm (Squeeze thm for sequences approaching infinity)

Let (x_n) (y_n) be two sequences

(1) Assume that $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$ $x_n \leq y_n$

i) if $\lim_{n \to \infty} x_n = +\infty$ then $\lim_{n \to \infty} y_n = +\infty$

ii) if $\lim_{n \to \infty} y_n = -\infty$ then $\lim_{n \to \infty} x_n = -\infty$

Proof Indeed let $\lim_{n\to\infty} x_n = +\infty$

and $x_n \leq y_n$ for any $n > N$.

then take any $C>0$ there exist

K s.t. $\forall n > K$ $x_n > C$

then if $K < N$ we can guarantee

that $y_n > C$ since $n > N$ for $n > N$: since $n > K$.

in this case $y_n \geq x_n > C$.

If $K \geq N$ then for any $n > K$, we get

$$y_n \geq x_n > G \quad \text{for any } n > k.$$

$\blacksquare$

~~Squeeze thm~~

12) Assume that $\lim_{n\to\infty} \frac{x_n}{y_n} = q$ with $q \in \mathbb{R}_{>0}$

(i) If $\lim_{n\to\infty} x_n = +\infty$ then $\lim_{n\to\infty} y_n = +\infty$

(ii) If $\lim_{n\to\infty} y_n = -\infty$ then $\lim_{n\to\infty} x_n = -\infty$